Solving nonlinear circuits with pulsed excitation by multirate partial differential equations

Andreas Pels[1,2], Johan Gyselinck[3], Ruth V. Sabariego[4], and Sebastian Schöps[1,2]
[1]Graduate School of Computational Engineering, Technische Universität Darmstadt, Germany
[2]Institut für Theorie Elektromagnetischer Felder, Technische Universität Darmstadt, Germany
[3]BEAMS Department, Université libre de Bruxelles, Belgium
[4]Department of Electrical Engineering, EnergyVille, KU Leuven, Belgium

Multirate Partial Differential Equations (MPDEs) have been successfully used in nonlinear high-frequency applications with largely separated time scales. In this digest MPDEs are applied to obtain an efficient solution for nonlinear low-frequency electrical circuits with pulsed excitation. The MPDEs are solved by a Galerkin approach and time discretization. Nonlinearities are efficiently accounted for by neglecting the high-frequency components (ripples) of the state variables and using only their envelope for the evaluation.

***Index Terms*—Finite element analysis, Nonlinear circuits, Numerical simulation, Partial differential equations.**

I. Introduction

Multirate Partial Differential Equations (MPDEs) have been successfully used in nonlinear high-frequency applications with largely separated time scales [1], [2], [3], [4]. The focus in this digest is the efficient solution of nonlinear low-frequency electrical circuits with pulsed excitation. The system of differential-algebraic equations (DAEs) describing a circuit is first rewritten as MPDEs. The MPDEs are solved by a Galerkin approach and time discretization. The recently introduced PWM (pulse width modulation) basis functions [5] are used for the solution expansion.

To efficiently account for nonlinearities, the high-frequency components of the state variables (ripples), due to the pulsed excitation, are neglected and the nonlinearity is evaluated using only the envelope of the state variables. This new approach is validated on the example of a simplified buck converter, see [5], using a nonlinear coil. The solution of this simplified buck converter as well as its excitation is depicted in Fig. 1. It consists of a fast periodically varying ripple and a slowly varying envelope.

II. Multirate formulation

To derive the multirate formulation we start from a generic DAE describing a nonlinear circuit

$$\mathbf{A}(\mathbf{x}(t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{x}(t) = \mathbf{c}(t), \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^{N_\mathrm{s}}$ is the vector of N_s state variables, $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}) \in \mathbb{R}^{N_\mathrm{s}\times N_\mathrm{s}}$ are matrices and $\mathbf{c}(t) \in \mathbb{R}^{N_\mathrm{s}}$ is the excitation vector. The DAEs are hereafter rewritten as MPDEs [1], [2] in terms of two time scales t_1 and t_2

$$\mathbf{A}(\widehat{\mathbf{x}})\left(\frac{\partial\widehat{\mathbf{x}}}{\partial t_1} + \frac{\partial\widehat{\mathbf{x}}}{\partial t_2}\right) + \mathbf{B}(\widehat{\mathbf{x}})\,\widehat{\mathbf{x}}(t_1,t_2) = \widehat{\mathbf{c}}(t_1,t_2)\,, \quad (2)$$

where $\widehat{\mathbf{x}}(t_1,t_2)$ and $\widehat{\mathbf{c}}(t_1,t_2)$ are the multivariate forms of $\mathbf{x}(t)$ and $\mathbf{c}(t)$. If $\widehat{\mathbf{c}}(t_1,t_2)$ fulfills the relation $\widehat{\mathbf{c}}(t,t) = \mathbf{c}(t)$ then the solution of the DAEs can be extracted from the solution $\widehat{\mathbf{x}}(t_1,t_2)$ of the MPDEs by $\mathbf{x}(t) = \widehat{\mathbf{x}}(t,t)$. We choose $\widehat{\mathbf{c}}(t_1,t_2)$ such that it only depends on the time scale t_2.

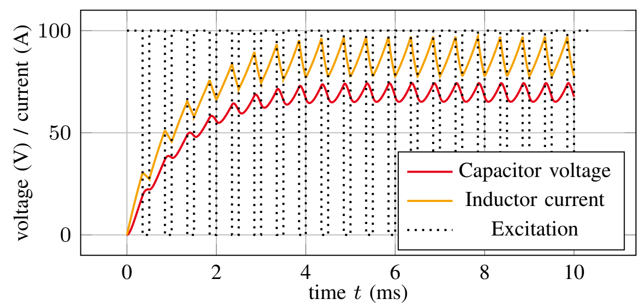

Fig. 1. Solution of nonlinear buck converter at $f_\mathrm{s} = 2000\,\mathrm{Hz}$. The switching cycle of the pulsed input voltage is $T_\mathrm{s} = 0.5\,\mathrm{ms}$, the duty cycle $D = 0.7$.

III. Solution of nonlinear MPDE

We denote the time scale t_1 as the slow time scale and the time scale t_2 as the fast time scale, which is assumed to be periodic. The fast time scale is discretized in a variational setting, i.e. using a Galerkin approach. We represent our solution by an expansion of $N_\mathrm{p}+1$ suitable basis functions $p_k(t_2)$ and coefficients $w_{j,k}(t_1)$. The approximated state variables $\widehat{\mathbf{x}}_j^h(t_1,t_2)$ can be written as

$$\widehat{\mathbf{x}}_j^h(t_1,t_2) = \sum_{k=0}^{N_\mathrm{p}} p_k(\tau)w_{j,k}(t_1) \text{ with } \tau = \frac{t_2}{T_\mathrm{s}} \bmod 1, \quad (3)$$

where T_s is the switching cycle of the excitation. Applying a Galerkin approach, i.e. testing (2) on T_s, yields

$$\int_{t_1-T_\mathrm{s}/2}^{t_1+T_\mathrm{s}/2} \Bigg[\mathbf{A}(\widehat{\mathbf{x}}^h)\left(\frac{\partial\widehat{\mathbf{x}}^h}{\partial t_1} + \frac{\partial\widehat{\mathbf{x}}^h}{\partial t_2}\right) + \mathbf{B}(\widehat{\mathbf{x}}^h)\,\widehat{\mathbf{x}}^h(t_1,t_2) - \widehat{\mathbf{c}}(t_1,t_2)\Bigg]\, p_l(\tau(t_2))\,\mathrm{d}t_2 = 0,\ \forall l \in [0,\dots,N_\mathrm{p}]. \quad (4)$$

The integration with respect to t_2 leads to DAEs in t_1, which can be solved by conventional time discretization. However in every time-step the integrals of (4) have to be reevaluated which leads to increased computational efforts. To circumvent this bottleneck we neglect the fast periodically varying ripples and only use the envelope for the evaluation of the nonlinearity.

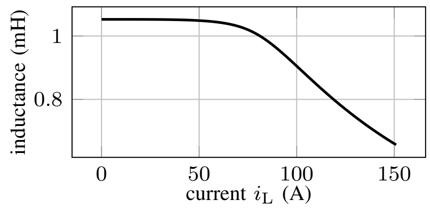

Fig. 2. Inductance of nonlinear coil versus current i_{L} through the coil.

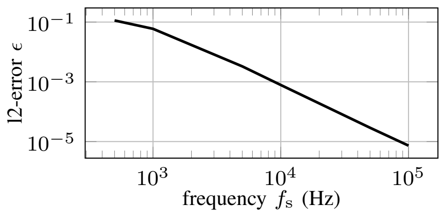

Fig. 3. Discrete l2-error ϵ of the MPDE approach versus frequency f_{s}.

According to (3) the envelope is stored in the vector of coefficients $\mathbf{w}(t_1)$. Therefore the matrices $\mathbf{A}$, $\mathbf{B}$ only depend on t_1 and the evaluation of (4) simplifies. Let $\mathbf{f}(\mathbf{w})$ be a function which extracts the envelope from $\mathbf{w}(t_1)$. Introducing

$$\boldsymbol{\mathcal{I}} \;=\; T_{\mathrm{s}} \int_0^1 \mathbf{p}(\tau)\, \mathbf{p}^{\top}(\tau)\, \mathrm{d}\tau\,, \tag{5}$$

$$\boldsymbol{\mathcal{Q}} \;=\; \int_0^1 \mathbf{p}(\tau)\, \frac{\mathrm{d}\mathbf{p}^{\top}}{\mathrm{d}\tau}\, \mathrm{d}\tau\,, \tag{6}$$

equation (4) becomes

$$\boldsymbol{\mathcal{A}}(\mathbf{w}(t_1))\,\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t_1} + \boldsymbol{\mathcal{B}}(\mathbf{w}(t_1))\,\mathbf{w}(t_1) \;=\; \boldsymbol{\mathcal{C}}(t_1)\,, \tag{7}$$

where

$$\boldsymbol{\mathcal{A}}(\mathbf{w}) = \mathbf{A}(\mathbf{f}(\mathbf{w})) \otimes \boldsymbol{\mathcal{I}}, \tag{8}$$

$$\boldsymbol{\mathcal{B}}(\mathbf{w}) = \mathbf{B}(\mathbf{f}(\mathbf{w})) \otimes \boldsymbol{\mathcal{I}} + \mathbf{A}(\mathbf{f}(\mathbf{w})) \otimes \boldsymbol{\mathcal{Q}}, \tag{9}$$

$$\boldsymbol{\mathcal{C}}(t_1) = \int_{t_1 - T_{\mathrm{s}}/2}^{t_1 + T_{\mathrm{s}}/2} \begin{bmatrix} \widehat{\mathbf{c}}_1(t_1, t_2)\, \mathbf{p}(\tau(t_2)) \\ \vdots \\ \widehat{\mathbf{c}}_{N_{\mathrm{s}}}(t_1, t_2)\, \mathbf{p}(\tau(t_2)) \end{bmatrix} \mathrm{d}t_2\,, \tag{10}$$

i.e. $\boldsymbol{\mathcal{A}}(\mathbf{w}), \boldsymbol{\mathcal{B}}(\mathbf{w}) \in \mathbb{R}^{N_{\mathrm{s}}(N_{\mathrm{p}}+1) \times N_{\mathrm{s}}(N_{\mathrm{p}}+1)}$, $\boldsymbol{\mathcal{C}}(t) \in \mathbb{R}^{N_{\mathrm{s}}(N_{\mathrm{p}}+1)}$. The DAEs (7) can be efficiently solved by conventional time discretization using much larger time-steps than the original DAEs (1). However, a drawback is the larger matrices.

IV. Numerical application

As basis functions $p_k(\tau)$ we choose the problem-specific PWM basis functions introduced by [5]. They are defined as piecewise polynomials starting from a piecewise linear function

$$p_1(\tau) \;=\; \begin{cases} \sqrt{3}\ \frac{2\tau - D}{D} & \text{if} \quad 0 \leq \tau \leq D \\ \sqrt{3}\ \frac{1 + D - 2\tau}{1 - D} & \text{if} \quad D \leq \tau \leq 1 \end{cases}, \tag{11}$$

The basis functions of higher order $p_k(\tau), 2 \leq k \leq N_{\mathrm{p}}$ are calculated by integrating $p_{k-1}(\tau)$ and orthonormalizing. The zero-th basis function is constant $p_0(\tau) = 1$ and the corresponding coefficient is used as envelope.

The numerical tests are performed on the simplified buck converter [5] using a nonlinear coil, which characteristic is shown in Fig. 2. For the solution expansion $N_{\mathrm{p}} = 4$ is used and (7) is solved using a high-order implicit Runge-Kutta method (Radau5, abstol $=$ reltol $= 10^{-6}$). The reference solution to which all results are compared is calculated directly by solving (1) also using Radau5 and very fine time discretization (abstol $=$ reltol $= 10^{-12}$). The buck converter is operated with different frequencies from $500\,\mathrm{Hz}$ to $100\,\mathrm{kHz}$. Fig. 3 depicts the error $\epsilon = \frac{\|v_{\mathrm{C,ref}} - v_{\mathrm{C}}\|_{l2}}{\|v_{\mathrm{C,ref}}\|_{l2}}$ of the MPDE approach with respect to the frequency $f_{\mathrm{s}} = \frac{1}{T_{\mathrm{s}}}$. As the magnitude of the ripples in relation to the envelope decreases with increasing frequency, the accuracy of the method rises. Table I shows the speedup (in terms of time for solving the equation systems) of the MPDE approach compared to conventional time discretization at same accuracy. For higher frequency conventional time discretization of (1) becomes more and more inefficient as a higher number of ripples has to be resolved. The MPDE approach on the contrary resolves the ripples with the Galerkin approach so that the time discretization resolves only the envelope. This leads to almost constant resolution time independent of the frequency.

TABLE I
Speedup of MPDE approach compared to conventional time discretization for different frequencies.

f_{s} / kHz	approx. speedup
10	33
50	207
100	529

V. Conclusion

The MPDE approach is applied to a nonlinear low-frequency example with pulsed excitation. The solution is obtained by a Galerkin approach and time discretization. To evaluate the nonlinearity the ripple components due to the pulsed excitation are neglected and only the envelope is used. The accuracy of the proposed method rises with increasing excitation frequency and the method offers a considerable speed-up compared to conventional time discretization with the same accuracy. The full paper will also consider a nonlinear field model.

Acknowledgement

This work is supported by the 'Excellence Initiative' of German Federal and State Governments and the Graduate School CE at TU Darmstadt and in part by the Walloon Region of Belgium (WBGreen FEDO, grant RW-1217703).

References

[1] H. G. Brachtendorf, G. Welsch, R. Laur, and A. Bunse-Gerstner, "Numerical steady state analysis of electronic circuits driven by multi-tone signals," *Electr. Eng.*, vol. 79, no. 2, pp. 103–112, 1996.

[2] J. Roychowdhury, "Analyzing circuits with widely separated time scales using numerical PDE methods," *IEEE Trans. Circ. Syst. Fund. Theor. Appl.*, vol. 48, no. 5, pp. 578–594, May 2001.

[3] K. Bittner and H. G. Brachtendorf, "Adaptive multi-rate wavelet method for circuit simulation," *Radioengineering*, vol. 23, no. 1, Apr. 2014.

[4] T. Mei, J. Roychowdhury, T. Coffey, S. Hutchinson, and D. Day, "Robust, stable time-domain methods for solving MPDEs of fast/slow systems," *IEEE Trans. Comput. Aided. Des. Integrated Circ. Syst.*, vol. 24, no. 2, pp. 226–239, Feb. 2005.

[5] J. Gyselinck, C. Martis, and R. V. Sabariego, "Using dedicated time-domain basis functions for the simulation of pulse-width-modulation controlled devices – application to the steady-state regime of a buck converter," in *Electromotion 2013*, Cluj-Napoca, Romania, Oct. 2013.